Equilibria in a Dynamic Model of Differentiated Duopoly with Demand Enhancing Investment

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Abstract
This paper examines a dynamic game of differentiated duopoly with demand enhancing investments in which one firm's investments have spillover effects on the demand for the rival firm's product. We consider two settings regarding firms' investment decisions, one in which the firms undertake investments noncooperatively and the other in which they choose investments cooperatively. We show that under each of the two settings, there exists a unique symmetric open-loop Nash equilibrium and that each firm's investment level is larger under the case of cooperative investment than under the case of noncooperative investment at open-loop Nash equilibrium. We also demonstrate that there exist stable linear Markov perfect equilibria. Furthermore, we examine non-linear Markov perfect equilibria.

Keywords: Differential games; Product differentiation; Bertrand competition; Spillovers

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1. Introduction

We study a dynamic model of a duopolistic industry in which one firm's investments have positive effects on demand for its product as well as demand for its rival firm's product. These investments might be thought of as research and development for new products or as advertisement. We assume such demand enhancing investments by one firm have spillover effects on the demand for the rival firm's product. For example, as R and D interpretation on investments, we could think of a situation in which an improvement in product quality by one firm can be quickly imitated by its rival firm. As advertisement interpretation, we may consider a situation in which consumers get product information about one product as well as its rival firm's product when the firms advertise their products at automobile shows. We construct a differential game to examine duopolistic firms' investments in a continuous time infinite horizon model. In the model, the oligopolistic firms may enhance demand for their products through their cumulative investments. Fershtman and Muller (1984) and Reynolds (1987) analyzed differential games on capital investment, but they did not consider spillovers effects.[2] We analyze a differential game to examine demand enhancing investment with spillovers in a differentiated duopoly. D'Aspremont and Jacquemin (1988) and Kamien et al. (1992) examined the effects of R and D spillovers on cooperative and noncooperative R and D in duopolistic markets in a static model. We consider two settings regarding firms' investment decisions. One setting is the case in which the firms make their decisions noncooperatively and the other is the one in which the firms can cooperate when they undertake investments. We assume that the firms engage in Bertarnd competition in the product markets.

For results, first we show that there exists a unique symmetric open-loop Nash equilibrium under both noncooperative investment and cooperative investment settings. Then we compare equilibrium investment levels and demonstrate that each firm's investment level under cooperation is larger than under the case of noncooperative investment. Next we examine feedback strategies and derive linear Markov perfect equilibrium. Furthermore we consider non-linear feedback strategies and demonstrate that there exist non-linear Markov perfect equilibria.

The paper is organized as follows. In Section 2, we introduce the basic model. In Section 3, we derive open-loop Nash equilibrium for games both under the case of noncooperative investment and under the case of cooperative investment. In Section 4, we examine Markov feedback strategies. First we consider linear Markov strategies and then examine non-linear Markov strategies. Section 5 concludes.

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[2]For differential games, see Basar and Olsder (1995), Kamien and Schwartz (1991), and Starr and Ho (1969).

2. The Model

There are two firms in the industry. The two firms produce differentiated products. Let q_i be firm i 's output, $i \quad I$, $I \quad 1,2$. Following Vives (1984), we assume a representative consumer's preference is given by

$$U \; q_1, q_2 \quad a \; q_1 \quad q_2 \quad \frac{1}{2} \; q_1^2 \quad q_2^2 \quad bq_1q_2.$$

The inverse demand functions of the products be given by

$$p_1 \quad a \quad q_1 \quad bq_2$$
$$p_2 \quad a \quad bq_1 \quad q_2,$$

where p_i is the price of product i , $a \quad 0,$ and $1 \quad b \quad 0$.
Then the demand functions are

$$q_1 \qquad p_1 \quad p_2$$
$$q_2 \qquad p_1 \quad p_2,$$

where

$$\frac{a \; 1 \quad b}{1 \quad b^2},$$
$$\frac{1}{1 \quad b^2},$$
$$\frac{b}{1 \quad b^2}.$$

We assume that the firms engage in Bertrand competition in the product markets.
The constant unit cost of production of firm i is c_i and for simplicity, we assume $c_i \quad c$ for $i \quad 1,2$. Let x_i denote the amount of investment by firm i . The firms' investments can be thought of as R and D or advertisement. We assume that the cost of investment is $\frac{1}{2}x_i^2$. Each firm may increase demand for its product through cumulative investment.
Let $y \; t$ be cumulative investment at time t . Let be a parameter representing decay or obsolescence regarding cumulative investment. Let us consider the setting in which one firm's investment has spillover effects on the other firm's demand. Specifically, we assume that changes in cumulative investment take the following form:

$$y \; t \quad \frac{dy \; t}{dt} \quad x_i \; t \quad x_j \; t \quad y \; t \; ,$$

where $\quad 0$ denotes a decay parameter and $y \; 0 \quad y_0$.

We assume that the demand intercept a at time t is given by

$$a \ t \quad a \quad y \ t \ ,$$

where a is the demand intercept at time zero and denotes an effectivity parameter of cumulative investment on demand for each product, 0 .
The firms are assumed to engage in Bertrand competition in the product market. Then Bertrand equilibrium prices p_1^B, p_2^B in the product market are

$$p_1^B \quad \frac{\quad c}{2} \quad p_2^B.$$

Thus firm i s instantaneous profit $_i$ is given by

$$_i \ a \ t \qquad \left[\frac{\quad c}{2}\right]^2, \ i, j \quad I, \ i \quad j.$$

The objective of each firm is to maximize the discounted sum of its instantaneous profits over an infinite time horizon. Thus the problem for firm i is to maximize

$$Z^i \quad _0\left[\ _i \ a \ t \qquad \frac{1}{2} x_i^2 \ t \ \right] e^{\ rt} dt, \ i, j \quad I, \ i \quad j,$$

subject to (2), (3), and $x_i \ t \quad 0$, where r is a common constant discount rate.

3. Open-loop Nash Equilibrium

In this section, first, we examine the game in which the firms decide their investments unilaterally so as to maximize their individual discounted sum of instantaneous profits over an infinite time horizon. Next we consider the game in which the firms coordinate their investment decisions so as to maximize their joint profits over an infinite time horizon. In this section, we consider open-loop strategies. Open-loop strategies and open-loop Nash equilibrium are defined as follows.
The open-loop strategy space for firm is the set

$$X_i \quad x_i \ t \ : x_i \ t \ \text{ is piecewise continuous and } x_i \ t \quad 0 \text{ for every } t \ .$$

An open-loop Nash equilibrium is an open-loop strategy selection $x \quad x_i, x_j$ such that for each $i \quad I$,

$$Z^i\ x_i\ ,x_j \qquad Z^i\ x_i,x_j\ \ , \text{ for } \quad x_i \quad X_i.$$

First, we consider the case where the firms choose investment strategies noncooperatively. For open-loop strategies, we have the following result.

Proposition 1: *There exists a unique stationary open-loop Nash equilibrium under the case of noncooperative investment. Each firm's equilibrium strategy is given by*

$$x^{BN} \qquad \frac{2\ a \quad c \quad 1 \quad b^{\ 2}}{r \qquad 2 \quad b^{\ 2} \quad 4\ 1 \quad b^{\ 2}},$$

and the equilibrium value of cumulative investment by

$$y^{BN} \qquad \frac{4\ a \quad c \quad 1 \quad b^{\ 2}}{r \qquad 2 \quad b^{\ 2} \quad 4\ 1 \quad b^{\ 2}}.$$

Proof For each $i \quad I,$ firm i 's problem is to maximize (8) subject to (2), (3), and the other firm's strategy $x_j\ t$. To find an open-loop Nash equilibrium, we need to solve all the problems faced by the firms simultaneously. Let the current-value Hamiltonian for firm i be

$$H_i \qquad {}_i\ a\ t \qquad \frac{1}{2}x_i^2\ t \qquad {}_i\ t\ \ x_i\ t \quad x_j\ t \qquad y\ t \ \ ,$$

where ${}_i$ is a costate variable, $i \quad I,\ i \quad j.$

The necessary conditions for an open-loop Nash equilibrium are, for $i \quad I,\ i \quad j,$

$$\frac{H_i}{x_i} \qquad x_i\ t \qquad {}_i\ t \qquad 0,$$

$$\begin{aligned} {}_i\ t \quad r\ {}_i\ t \quad & \frac{H_i}{y} \\ & r \qquad {}_i\ t \quad 2\ a \quad c \left[\frac{\qquad c}{2}\right]^2 \end{aligned} \quad ,$$

and

$$\lim_{t} e^{\ rt}\ {}_i\ t \qquad 0.$$

We shall show that the open-loop Nash equilibrium strategies are symmetric. Solving (13) and using the transversality condition yields

$${}_i \qquad {}_t\, e^{\;r} \qquad {}_t\, \frac{2\;a\quad c\quad 1\quad b^{2}}{2\quad b^{2}}\, d\;.$$

Since the right-hand side of (15) does not depend on i, we obtain

$${}_i \qquad {}_j,\; i, j \quad I,\; i \quad j.$$

It follows from (12) that $x_i \quad x_j, i, j \quad I,\; i \quad j.$
At the steady state, we must have ${}_i \quad 0$ and $y \quad 0$. Then

$$r \qquad {}_i \quad \frac{2\;a\quad c\quad 1\quad b^{2}}{2\quad b^{2}}$$

and

$$x_i \quad x_j \quad y \quad 0.$$

Thus at the symmetric equilibrium, we have

$$y \quad \frac{2}{\;}x$$

and

$$x \quad \frac{2\;a\quad c\quad 1\quad b^{2}}{r\qquad 2\quad b^{2}\quad 4\;1\quad b^{2}}.$$

Hence we have the steady state equilibrium investment given by

$$x^{BN} \quad \frac{2\;a\quad c\quad 1\quad b^{2}}{r\qquad 2\quad b^{2}\quad 4\;1\quad b^{2}}$$

and the equilibrium value of cumulative investment by

$$y^{BN} \quad \frac{4\;a\quad c\quad 1\quad b^{2}}{r\qquad 2\quad b^{2}\quad 4\;1\quad b^{2}}.$$

For the effects of changes in parameter values on the equilibrium investment, we note that
Proposition 2: The investment level at the steady state equilibrium is (i) decreasing in and (ii) decreasing in b, and (iii) increasing in .
Proof. From (9) and (10), we have

$$\frac{x^{BN}}{\;} \quad 0,$$

$$\frac{x^{BN}}{b} \quad 0,$$

and

$$\frac{x^{BN}}{} \quad 0.$$

The following proposition describes the stable open-loop Nash equilibrium state trajectory.
Proposition 3: *The open-loop Nash equilibrium state trajectory is given by*

$$y \; t \quad y^D \quad y_0 \quad y^D \;\; e^{\;_2 t},$$

where

$$_2 \quad \frac{r \quad \sqrt{r^2 \quad 4E}}{2}, \quad E \quad\quad r \quad\quad \frac{4 \; 1 \quad b^{\;2}}{2 \quad b^{\;2}},$$

$$F \quad \frac{4 \; \bar{a} \quad c \quad 1 \quad b^{\;2}}{2 \quad b^{\;2}}, \quad \textit{and} \;\; y^D \quad \frac{E}{F}.$$

Proof Differentiating (2) with respect to time yields

$$y \quad x_i \quad x_j \quad y,$$

where $y \quad \frac{d^2y}{dt^2}$.
It follows from (5), (13), and (16) that we have the following second order differential equation:

$$\begin{aligned} y \quad 2 \; r \quad x \; \frac{4 \; a \quad c \quad 1 \quad b^{\;2}}{2 \quad b^{\;2}} \quad y \\ r \quad y \quad y \; \frac{4 \; a \quad c \quad 1 \quad b^{\;2}}{2 \quad b^{\;2}} \quad y \\ ry \quad \left[\; r \quad \frac{4 \; 1 \quad b^{\;2}}{2 \quad b^{\;2}} \right] y \quad \frac{4 \; \bar{a} \quad c \quad 1 \quad b^{\;2}}{2 \quad b^{\;2}}. \end{aligned}$$

We can rewrite (17) as

$$y \quad ry \quad Ey \quad F \quad 0,$$

where

$$E \quad r \quad \frac{4\ 1 \quad b^{\ 2}}{2 \quad b^{\ 2}} \text{ and } F \quad \frac{4\ \bar{a} \quad c \quad 1 \quad b^{\ 2}}{2 \quad b^{\ 2}}.$$

A particular solution of this differential equation is

$$y\ t \quad \frac{E}{F} \quad y^D.$$

Note that this particular solution is exactly the steady state of the open-loop Nash equilibrium derived above.
Then the roots of the characteristic equation associated with the homogeneous part of (18) are given by

$$_1 \quad \frac{r \quad \sqrt{r^2 \quad 4E}}{2}$$

and

$$_2 \quad \frac{r \quad \sqrt{r^2 \quad 4E}}{2}.$$

For a stable equilibrium, we take $\ _2\ $.
Therefore the stable open-loop Nash equilibrium trajectory is

$$y\ t \quad y^D \quad y_0 \quad y^D \quad e^{\ _2 t}.$$

Next we analyze the case where the two firms can cooperate when they make their investment decisions while they compete in the product market. Then the problem for the firms is to choose their investments to maximize

$$Z^c \quad \sum_{i\ 1}^{2} \int_0 \left[\ _i\ a\ t\ , c \quad \frac{1}{2} x_i^2\ t\ \right] e^{\ rt} dt$$

subject to (2), (3), and $x_i\ t \quad 0\ , \quad i \quad I$.
For the case of cooperative investment, we have the following result.
Proposition 4: *There exists a unique stationary open-loop Nash equilibrium under the case of cooperative investment. Each firm's equilibrium strategy is given by*

$$x^{BC} \quad \frac{4\ a \quad c \quad 1 \quad b^{\ 2}}{r \quad 2 \quad b^{\ 2} \quad 4\ 1 \quad b^{\ 2}}$$

and the equilibrium value of cumulative investment by

$$y^{BC} \quad \frac{8\ a \quad c \quad 1 \quad b\ ^2}{r \qquad 2 \quad b\ ^2 \quad 4\ 1 \quad b\ ^2}.$$

Proof Let the current-value Hamiltonian in this case be

$$\sum_{i\ 1}^{2} \left[\ _i\ a\ t \qquad \frac{1}{2} x_i^2\ t \ \right] \qquad t \quad x_1\ t \quad x_2\ t \qquad y\ t \quad ,$$

where is a costate variable. The necessary conditions for an open-loop Nash equilibrium are, for $i \quad I$,

$$\frac{}{x_i} \qquad x_i \qquad 0,$$

$$r \qquad \frac{}{y}$$

$$r \qquad \frac{4\ \bar{a} \quad c \quad 1 \quad b\ ^2}{2 \quad b\ ^2},$$

and

$$\lim_{t} e^{\ rt} \qquad 0.$$

At the steady state, we must have and $y \quad 0$ and $\quad 0$. Thus we get

$$y \quad \frac{2}{}x$$

and

$$r \qquad \frac{4\ \bar{a} \quad c \quad 1 \quad b\ ^2}{2 \quad b\ ^2}.$$

It follows from these equations that at the steady state, we obtain

$$x^{BC} \quad \frac{4\ a \quad c \quad 1 \quad b\ ^2}{r \qquad 2 \quad b\ ^2 \quad 4\ 1 \quad b\ ^2}$$

and

$$y^{BC} \quad \frac{8\ a \quad c \quad 1 \quad b\ ^2}{r \qquad 2 \quad b\ ^2 \quad 4\ 1 \quad b\ ^2}.$$

Now we can compare the investment level at noncooperative equilibrium with the one at cooperative equilibrium. Recall that we have, for the noncooperative investment case,

$$x^{BN} \quad \frac{2\ a \quad c \quad 1 \quad b^{2}}{r \quad 2 \quad b^{2} \quad 4\ 1 \quad b^{2}}$$

and for the cooperative investment case,

$$x^{BC} \quad \frac{4\ a \quad c \quad 1 \quad b^{2}}{r \quad 2 \quad b^{2} \quad 4\ 1 \quad b^{2}}.$$

Proposition 5: *At the open-loop Nash equilibrium, each firm's investment level is larger under the cooperative investment case than under the noncooperative investment case, that is,*

$$x^{BN} \quad x^{BC}.$$

Proof: Evident from (9) and (23).

4. Markov perfect equilibrium

In the previous section, we considered open-loop strategies. In this section, we examine feedback strategies. First we look at liner Markov feedback strategies. Next we shall derive Markov perfect equilibrium with nonlinear feedback strategies. To consider nonlinear feedback strategies, we shall follow the method in Tsutsui and Mino (1990). Feedback strategies and Markov perfect equilibrium are defined as follows. The feedback strategy space for player i is the set

$$X_i^F \quad x_i\ y,t \ \text{ is continuous in } \ y,t\ ,\ x_i\ y,t \quad 0, \text{ and } y \quad 0\ .$$

A Markov perfect equilibrium is a pair of feedback strategies x_i, x_j such that for each $i \quad I$,

$$Z^i\ x_i, x_j \qquad Z^i\ x_i, x_j \ \text{ for every } x_i \quad X_i^F.$$

For linear Markov strategies, we have the following result.
Proposition 5: *There exist linear Markov perfect equilibria given by*

$$x_i \quad K \quad Ly$$

where

$$L \quad \frac{r \quad 2 \qquad \sqrt{r \quad 2 \quad ^2 \quad 24\left[\frac{a \ c \ 1 \ b}{2 \ b}\right]^2}}{6}$$

and

$$K \quad \frac{\left[\frac{\bar{a} \ c \ 1 \ b}{2 \ b}\right]^2}{r \qquad 3L}.$$

Proof Let $V^i \ y$ be the value function for firm i. Then feedback Nash equilibrium strategies must satisfy a system of Hamilton-Jacobi-Bellman equation, for $i \quad I$,

$$rV^i \ y \qquad \max_{x_i}\left\{ \ _i \ a \qquad \frac{1}{2}x_i^2 \quad \frac{dV^i \ y}{dy} \ x_i \quad x_j \qquad y \ \right\}$$

where

$$a \qquad \left[\frac{a \quad c \quad 1 \quad b}{2 \quad b}\right]^2.$$

Solving the maximization problem of the right-hand side of (27) yields

$$x_i \qquad \frac{dV^i \ y}{dy}.$$

We assume that the value function is symmetric. Suppose that the value function takes the following form:

$$V \ y \qquad J \quad Ky \quad \frac{1}{2}Ly^2.$$

Then

$$x_i \qquad K \quad Ly.$$

Substituting (31) and (32) into (30) results in

$$r\left[J \quad Ky \quad \frac{1}{2}Ly^2\right]$$
$$\left[\frac{\bar{a} \quad y \quad c \quad 1 \quad b}{2 \quad b}\right]^2 \quad \frac{1}{2} \; K \quad Ly \;\; ^2$$
$$K \quad Ly \quad K \quad Ly \quad K \quad Ly \quad\quad y \; .$$

This equation must hold for any y, and hence we must have

$$\frac{1}{2}rL \quad \frac{3}{2}L^2 \quad\quad L \quad \left[\frac{1 \quad b}{2 \quad b}\right]^2 \quad\quad 0,$$

$$rK \quad 3KL \quad\quad K \quad 2 \; \bar{a} \quad\quad c \quad \left[\frac{1 \quad b}{2 \quad b}\right]^2 \quad\quad 0,$$

and

$$rJ \quad \left[\frac{a \quad c \quad 1 \quad b}{2 \quad b}\right]^2 \quad \frac{3}{2}K^2 \quad\quad 0.$$

It follows from (34), (35) and (36) that we obtain

$$L \quad\quad \frac{r \quad 2 \quad\quad \sqrt{r \quad 2 \quad ^2 \quad 24\left[\frac{1 \; b}{2 \; b}\right]^2}}{6},$$

$$K \quad\quad \frac{2 \; \bar{a} \quad\quad c \quad \left[\frac{1 \; b}{2 \; b}\right]^2}{r \quad\quad\quad 3L},$$

and

$$J \quad\quad \frac{1}{r}\left[\left[\frac{a \quad c \quad 1 \quad b}{2 \quad b}\right]^2 \quad\quad \frac{3}{2}K^2\right].$$

Substituting (32) into (2) yields

$$y \quad\quad 2L \quad\quad\quad y \quad 2K \quad\quad 0.$$

A particular solution to the differential equation (40) is

$$y \qquad \frac{2K}{2L}.$$

The solution to the homogeneous part of (40) is

$$y\ t \qquad M\, e^{\quad 2G\ \ t},$$

where M is a constant. Therefore the complete solution of (40) is

$$y\ t \qquad y \qquad y_0 \quad y\ \ e^{\quad 2G\ \ t}.$$

For this state trajectory to be asymptotically stable, we must have

$$2L.$$

Next we look at non-linear Markov strategies. Following Tsutsui and Mino (1990), we derive an auxiliary equation from the Hamilton-Jacobi-Bellman equation and then solve that auxiliary equation. Then we derive a solution of the HJB equation from a solution of the auxiliary equation.

Proposition 6: *There exists a Markov perfect equilibrium given by*

$$x_i \qquad k\ y$$

and

$$V^i\ y \qquad\qquad y$$

for each y *satisfying* (61), *where* $k\ y$ *is a solution to* (44) *below, and* y *is defined by* (59).

Proof. Recall that the Hamilton-Jacobi-Bellman equation is given by

$$rV^i\ y \qquad \max_{x_i}\left\{\ {}_i\ a \qquad \frac{1}{2}x_i^2 \quad \frac{dV^i\ y}{dy}\ x_i \quad x_j \qquad y\ \right\}.$$

Solving the maximization problem on the right-hand side of (30) yields

$$x_i \qquad \frac{dV^i\ y}{dy}.$$

Let $V_y^i \quad \frac{dV^i\ y}{dy}.$ Substituting (43) into (30) results in

$$rV^i \quad \left[\frac{a \quad c \quad 1 \quad b}{2 \quad b}\right]^2 \quad \frac{1}{2}\ V_y^i\ {}^2 \quad V_y^i\ V_y^i \quad V_y^j \quad y\ .$$

Differentiation of (44) with respect to y yields

$$rV_y^i \quad 2\ a \quad c \quad y \left[\frac{1 \quad b}{2 \quad b}\right]^2 \quad V_y^i V_{yy}^i \quad 2V_y^i V_{yy}^i \quad V_{yy}^i V_y^j \quad V_{yy}^j V_y^i \quad V_y^i \quad yV_{yy}^i\ ,$$

where $V_{yy}^i \quad \frac{d^2V^i\ y}{dy^2}$.

Now we assume symmetry, that is, $V_y^i \quad V_y^j \quad V_y$. Let $z \quad V_y$ and $z_y \quad V_{yy}$.
Then the HJB equation can be rewritten as

$$\begin{aligned} rz \quad & 2\ a \quad c \quad y \left[\frac{1 \quad b}{2 \quad b}\right]^2 \quad zz_y \quad z_y\ 2z \quad y \quad z\ 2z_y \\ & 2\ a \quad c \quad y \left[\frac{1 \quad b}{2 \quad b}\right]^2 \quad 3zz_y \quad z_y y \quad z. \end{aligned}$$

Thus we obtain

$$z_y \quad \frac{r \quad z \quad 2\ a \quad c \quad y \left[\frac{1\ b}{2\ b}\right]^2}{3z \quad y}.$$

Let $Y \quad y$ and $Z \quad z$. Then equation (47) can be written as

$$\begin{aligned} \frac{dZ}{dY} \quad & \frac{r \quad z \quad 2\ a \quad c \quad y \left[\frac{1\ b}{2\ b}\right]^2}{3z \quad y} \\ & \frac{r \quad Z \quad 2\ a \quad c \left[\frac{1\ b}{2\ b}\right]^2 \quad 2\ Y \left[\frac{1\ b}{2\ b}\right]^2}{3\ Z \quad Y} \\ & \frac{r \quad Z \quad 2Y\left[\frac{1\ b}{2\ b}\right]^2 \quad r \quad 2\ a \quad c \left[\frac{1\ b}{2\ b}\right]^2 \quad 2\left[\frac{1\ b}{2\ b}\right]^2}{3Z \quad 3 \quad Y} \\ & \frac{r \quad Z \quad 2Y\left[\frac{1\ b}{2\ b}\right]^2}{3Z \quad Y}. \end{aligned}$$

Here and must satisfy $3 \quad 0$ and

$r \quad 2 \; a \quad c \left[\frac{1 \; b}{2 \; b}\right]^2 \quad 2 \left[\frac{1 \; b}{2 \; b}\right]^2 \quad 0$. Hence and are given as

$$\frac{2 \; a \quad c}{3 \quad r \qquad 2}$$

and

$$\frac{2 \quad a \quad c}{9 \quad r \qquad 6}.$$

Now equation (48) can be written as

$$\frac{dZ}{dY} \quad \frac{r \qquad Z \quad 2Y\left[\frac{1 \; b}{2 \; b}\right]^2}{3Z \qquad Y}$$
$$\frac{r \qquad \frac{Z}{Y} \quad 2\left[\frac{1 \; b}{2 \; b}\right]^2}{3\frac{Z}{Y}}$$
$$\left(\frac{Z}{Y}\right).$$

Let $W \quad \frac{Z}{Y}$. Then equation (51) can be rewritten as

$$W \qquad W \quad \frac{dW}{dY}Y.$$

Furthermore we can rewrite equation (52) as

$$\frac{dY}{Y} \qquad \frac{dW}{W \qquad W}.$$

Then the solution to this differential equation is

$$W \quad W_A \qquad W \quad W_B \qquad QY,$$

where Q is a constant for integration, and W_A and W_B are the solutions of the equation $3W^2 \quad r \quad 2 \quad W \quad 2\left[\frac{1 \; b}{2 \; b}\right]^2 \quad 0.$ Assume $W_A \quad W_B$. Thus the solution to differential equation (53) can be rewritten as

$$z \qquad W_A Y \;^f\; z \qquad W_B Y \;^g \quad Q,$$

where

$$f \quad \frac{1 \quad \sqrt[3]{} W_A}{W_B \quad W_A}$$

and

$$g \quad \frac{\sqrt[3]{} W_B \quad 1}{W_B \quad W_A}.$$

The two singular solutions of equation (53) are

$$z \qquad W_A \; y$$

and

$$z \qquad W_B \; y \qquad .$$

Let $k \; y$ denote a solution of (44). For each $k \; y$, let us define y by

$$y \quad \frac{1}{r}\left[\; \bar{a} \quad c \quad y^{\,2}\left[\frac{1 \quad b}{2 \quad b}\right]^2 \; \frac{3}{2} \; k \; y^{\;\;2} \quad yk \; y \; \right].$$

Then y is a twice differentiable function defined on the domain of $k \; y$ and satisfies the HJB equation.

Next let us define $x \quad k \; y$. At a steady state y , $k \; y \quad \overline{2} y$.

Let $z \quad k \; y$. Linearizing $y \quad 2k \; y \quad y$ around the steady state y , we have

$$y \quad y \quad y \quad 2k \quad y \qquad .$$

Then the local stability condition of a given steady state y is given as $k \; y \quad \overline{2}$.

Thus y is locally stable if

$$z_y \; y \qquad \frac{r \quad \frac{y}{2} \quad 2 \; \bar{a} \quad c \quad y \quad \left[\frac{1 \; b}{2 \; b}\right]^2}{3\frac{y}{2} \quad y}$$

$$\frac{r \quad \frac{y}{2} \quad 2 \; \bar{a} \quad c \quad y \quad \left[\frac{1 \; b}{2 \; b}\right]^2}{\frac{y}{2}}$$

$$\overline{2}.$$

Hence we obtain

$$y\left(r \quad 4\ ^2\left[\frac{1 \quad b}{2 \quad b}\right]^2 \quad \frac{\ ^2}{2}\right) \quad 4\ \bar{a} \quad c\ \left[\frac{1 \quad b}{2 \quad b}\right]^2.$$

Therefore y is locally stable if (61) holds.
Thus we have established the existence of Markov perfect equilibrium with non-linear strategies.
Note that the two singular solutions of equation (53) derived above correspond to the linear Markov perfect equilibrium strategies obtained in Theorem 9.
Next we examine feedback strategies for the case of cooperative investment. In this case, the two firms maximize

$$\sum_{i\ 1}^{2} \int_0 \left[\ _i\ a\ t \quad \frac{1}{2}x_i^2\ t\ \right] e^{\ rt} dt$$

subject to (2), (3), and $x_i\ t \quad 0.$
Then the current-value Hamiltonian in this case is given by

$$\sum_{i\ 1}^{2} \left[\ _i\ a\ t \quad \frac{1}{2}x_i^2\right] \quad x_1 \quad x_2 \quad y\ ,$$

where is a costate variable.
Since the firms can coordinate investment strategies under cooperative investment scenario, Markov perfect equilibrium in this case is the same as the open-loop Nash equilibrium that was derived in Section 3.

5. Conclusion

In this paper, we have studied a dynamic game of differentiated duopoly with demand enhancing investment in which one firm's investment has spillover effects on the rival firm's demand for its product. We have examined two settings regarding firms' investment decisions, one in which the firms undertake investments noncooperatively and the other in which they choose investment cooperatively.
We have shown that for each game, there exists a unique symmetric open-loop Nash equilibrium. We have demonstrated that each firm's investment level is larger under the case of cooperative investment than under the case of noncooperative investment at open-loop Nash equilibrium. We have also demonstrated that there exist stable linear Markov perfect equilibria. Furthermore, we have shown that there exist non-linear Markov perfect equilibria.

References

Basar, T and G. J. Olsder (1995) Dynamic Noncooperative Game Theory, Academic Press, New York.

D'Aspremont, C and Jacquemin (1988) Cooperative and Noncooperative R and D in Duopoly with Spillovers, American Economic Review, 1133-1137.

Fershtman, C. and E. Muller (1984) ``Capital Accumulation Games of Infinite Duration'', Journal of Economic Theory, 38, 167-177.

Kamien, M. and N. Schwartz (1991) Dynamic Optimization, New York, Elsevier North Holland.

Kamien, M., E. Muller and I. Zang (1992) Research Joint Ventures and R and D Cartels, American Economic Review, 1293-1306.

Reynolds, S. (1987) ``Capacity Investment, Preemption and Commitment in an Infinite Horizon Model'', International Economic Review, 28, 67-88.

Starr, A.W. and Y. C. Ho (1969a) ``Nonzero-Sum Differential Games'', Journal of Optimization Theory and Applications, 3, 184-206.

Tsutsui, S. and K. Mino (1990) Nonlinear Strategies in Dynamic Duopolistic Competition with Sticky Prices, Journal of Economic Theory 52, 136-161.

Vives, X. (1984) Duopoly Information Equilibrium: Cournot and Bertrand'', Journal of Economic Theory 34, 71-94.